BFT embedding of the Green-Schwarz superstring and the pure spinor formalism

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Abstract

We worked out the Batalin-Fradkin-Tyutin (BFT) conversion program of second class constraints to first class constraints in the GS superstring using light cone coordinates. By applying this systematic procedure we were able to obtain a gauge system that is equivalent to the recent model proposed in [1] to relate the GS superstring to the pure spinor formalism.

1 Introduction

The covariant quantization of GS superstring is an important open problem for string theory and for the general theory of constrained systems. The problem is as old as the initial proposal of the classical action by Green and Schwarz [2], and many different ways to tackle it with a wide spectrum of techniques was worked along the years. Among them are covariant conversion procedures (with an infinite number of auxiliary fields), BRST program with infinitely reducible constraints [3], the use of light cone coordinates and the question of conformal invariance [4], and recently the pure spinor formalism [5], gauging cosets [6], and BRST extensions attempting to lift the bosonic spinor constraint of the pure spinor formalism by introducing more ghost variables but a finite number of them [7].

The problem with the quantization of the GS superstring lies in the fact that we do not know a procedure to separate the first and second class fermionic constraints in a manifestly Lorentz invariant way. The first class sector of these fermionic constraints is responsible for the κ symmetry of the superstring while the second class constraints appear as in any other fermionic system because

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the Lagrangian is linear in the time derivatives of the fermionic variables. So we face the problem of trying to quantize the system without splitting the constraints or split them in a non-covariant way and use a somewhat complicated Dirac bracket to perform the quantization. The recent proposed pure spinor formalism is an important step in the direction of a covariant quantization program for the GS superstring. This formulation is covariant but the price to pay is a radical departure from the BRST standard techniques, introducing a constraint in the bosonic ghost sector of the theory, know as the pure spinor constraint. The idea could be related to the use of all the constraints (not only the first class constraints) to construct the BRST operator. The condition $Q^2 = 0$ implies then a constraint in the ghost sector. Nevertheless, this quantization program has been proved to be very useful in many calculations that do not imply the explicit solution of the constrained ghost relation. The other standard formalism to describe the supersymmetric string, the RNS model, has the supersymmetry realized in the worldsheet so the space time supersymmetry is not manifest. As a consequence of this fact the spectrum of the RNS theory is not Lorentz covariant. We need to impose a projection of states in its spectrum, the so called GSO projection to recover a Lorentz invariant spectrum. Without a manifest space time supersymmetric action is very difficult -if not impossible- to describe the superstring in Ramond-Ramond backgrounds.

In what follows we will present a solution to the long standing problem of the implementation of the conversion program of second class constraints into first class constraints by adding to the GS action new fermionic variables with a standard symplectic structure. Our result is completely equivalent to the recent model proposed by Berkovits and Marchioro (BM) to relate the GS superstring with the pure spinor formalism [1]. From our point of view, the key observation given by these authors is that in the extended phase space the Lorentz generators close in the corresponding Lorentz algebra up to and exact BRST term. Moreover, and in spite of the non Lorentz covariant approach, the quantization program can be implemented because the anomaly that comes from the nonlinear products of the new added variables can be canceled using standard BRST techniques. A previous attempt to relate the pure spinor formalism with the GS superstring using standard BRST approach is [8] where the authors were able to modify the GS original action with a non-local term, that when added to the GS action, render it BRST invariant.

In our approach we do not need to fix the gauge (the semi-light cone gauge) nor to add variables other than the ones required by the conversion procedure. Our work is in close relation with a recent proposed action given in [9] where the authors start from a modified GS action doubling the fermion sector and introducing an interaction between the fermionic sectors by hand. These authors claim that theirs modified GS action is equivalent to the BM action by fixing the gauge and performing a Darboux transformation that changes the remaining Dirac bracket to a standard symplectic structure. We will see that this model can be explained and simplified using the conversion approach presented here.

In section 2 we will review the basic ideas of the BFT program. Section 3

will be used to expose our main results and the GS gauged action. In section 4 we present some comments on the relation between our results and the Aisaka-Kazama (AK) action [9], and in section 5 the conclusion and some notes about possible future work.

2 Birds eye to BFT procedure

The aim of the BFT method [10] is to develop a systematic approach to the conversion of a general set of constraints into an algebra of first class constraints by adding to the original phase space an appropriate number of new variables with its own symplectic structure. The method provides us with a procedure for the conversion of second class constraints into first class ones in this extended phase space and a procedure to modify any observable, including any previous first class constraint in such a way that we can construct an effective first class gauge algebra and an effective action consistent with the whole conversion procedure. It is based on homological perturbation theory and is, in this respect, very similar to the iterative construction of the BRST charge given a gauge algebra.

Suppose that we have a set of constraints where some constraints are second class χ_α, $\{\chi_\alpha, \chi_\beta\} = C_{\alpha\beta}$ and some are first class ϕ_m in a phase space defined by the coordinates z^i with the standard symplectic form σ^{ij}. Now we will add ξ_α new variables with symplectic structure $\omega^{\alpha\beta}$ to the original phase space. The idea is to construct a new set of constraints $\tilde{\chi}_\alpha$ satisfying the algebra

$$\{\tilde{\chi}_\alpha, \tilde{\chi}_\beta\} = 0. \tag{1}$$

To solve for $\tilde{\chi}_\alpha$ we propose a solution in power series of the new variables

$$\tilde{\chi}(z, \xi) = \sum_n X_\alpha^{(n)}, \tag{2}$$

where the $n = 0$ term coincides with the original constraint χ_α and $X_\alpha^{(n)}$ is a term proportional to ξ^n in the series expansion. The solution, up to a canonical transformation, in the extended phase space is [10]

$$X_\alpha^{(0)} = \chi_\alpha, \quad X_\alpha^{(1)} = X_{\alpha\gamma}\xi^\gamma, \quad X_{\alpha\gamma}\omega^{\gamma\delta}X_{\beta\delta} = -C_{\alpha\beta}(z), \tag{3}$$

and for the next terms $n \geq 2$ in the power series

$$X_\alpha^{(n+1)} = -\frac{1}{n+2}\xi^\beta\omega_{\beta\gamma}X^{\gamma\rho}X_{\rho\alpha}^{(n)}, \tag{4}$$

where

$$X_{\alpha\beta}^{(1)} = \{\chi_{[\alpha}, X_{\beta]}^{(1)}\}, \quad X_{\alpha\beta}^{(n)} = \sum_{m=0}^{n}\{X_\alpha^{(n-m)}, X_\beta^{(m)}\} + \sum_{m=0}^{n-2}\{X_\alpha^{(n-m)}, X_\beta^{(m+2)}\}_\xi, \tag{5}$$

and the first bracket is evaluated using only the original phase space variables and the second using only the new variables.

The same idea works also to extend any function $f(z)$ of the original variables z to a new function $\tilde{f}(z,\xi)$ as a solution in power series of the new variables ξ. This series must satisfy the condition

$$\{\tilde{\chi}_\alpha, \tilde{f}\} = 0, \quad \tilde{f} = \sum_n F^{(n)}, \tag{6}$$

where $F^{(0)} = \tilde{f}(z,0) = f(z)$ and $F^{(n)}$ is the term proportional to ξ^n. The solution is [10]

$$F^{(n+1)} = -\frac{1}{n+1}\xi^\beta \omega_{\beta\gamma} X^{\gamma\rho} F_\rho^{(n)}, \tag{7}$$

where

$$F_\alpha^{(0)} = \{\chi_\alpha, f(z)\}, \quad F_\alpha^{(1)} = \{X_\alpha^{(1)}, f(z)\} + \{\chi_\alpha, F^{(1)}\} + \{X_\alpha^{(2)}, F^{(1)}\}_\xi, \tag{8}$$

and

$$F_\alpha^{(n)} = \sum_{m=0}^{n} \{X_\alpha^{(n-m)}, F^{(m)}\} + \sum_{m=0}^{n-2} \{X_\alpha^{(n-m)}, F^{(m+2)}\}_\xi + \{X_\alpha^{(n+1)}, F^{(1)}\}_\xi. \tag{9}$$

In particular, we can extend the original first class constraints ϕ_m to a new set of constraints $\tilde{\phi}_m$ in such a way that all the new constrains close in a new gauge algebra in the extended phase space. In what follows we will need only terms up to second order in the new variables.

An interesting corollary of the conversion approach is that the original Dirac bracket can be recovered using

$$\{\tilde{A}, \tilde{B}\}|_{\xi=0} = \{A, B\}_D, \tag{10}$$

for any two functions of the original phase space A, B that were extended to $\tilde{A}, \tilde{B}$ as can be easily checked.

3 BFT embedding of the GS superstring

We start from the GS action that we write in the form

$$S = -\frac{1}{2}\int d^2\zeta \Big[\sqrt{-g}g^{ij}\Pi_i^\mu \Pi_{\mu j} + 2\varepsilon^{ij}\Pi_i^\mu (W_{j\mu}^1 - W_{j\mu}^2) - 2\varepsilon^{ij} W_i^{1\mu} W_{j\mu}^2\Big], \tag{11}$$

where

$$W_i^{A\mu} = i\theta^A \gamma^\mu \partial_i \theta^A, \quad \Pi_i^\mu = \partial_i x^\mu - \sum_A W_i^{A\mu}. \tag{12}$$

As usual the bosonic constraints can be obtained by setting to zero the energy-momentum tensor. Taking the conformal gauge, the first order Lagrangian associated is

$$\mathcal{L} = \dot{x}^\mu p_\mu + \dot{\theta}_\alpha^A p_\alpha^A - H_c - \lambda_\alpha^A d_\alpha^A, \tag{13}$$

where

$$p_\mu = \Pi_{0\mu} - (W_{1\mu}^1 - W_{1\mu}^2), \tag{14}$$

and

$$d^1_\alpha = p^1_\alpha - i(\theta^1\gamma^\mu)_\alpha(p_\mu - x'_\mu + W^1_{1\mu}), \tag{15}$$

$$d^2_\alpha = p^2_\alpha - i(\theta^2\gamma^\mu)_\alpha(p_\mu + x'_\mu - W^2_{1\mu}), \tag{16}$$

are the fermionic constraints. The canonical Hamiltonian is

$$H_c = \frac{1}{2}\Big[\Big(p_\mu + W^1_{1\mu} - W^2_{1\mu}\Big)^2 + \Big(x'_\mu - \sum_A W^A_{1\mu}\Big)^2\Big] = \frac{1}{2}\Big(\Pi^2_0 + \Pi^2_1\Big). \tag{17}$$

The bosonic constraints

$$\mathcal{H} = \frac{1}{2}\Big(\Pi^2_0 + \Pi^2_1\Big), \quad \mathcal{H}_1 = \Pi^\mu_0\Pi^\mu_1, \tag{18}$$

can be written in the form

$$\hat{T} = \mathcal{H} + \mathcal{H}_1 = \frac{1}{2}\hat{\Pi}^2, \quad T = \mathcal{H} - \mathcal{H}_1 = \frac{1}{2}\Pi^2, \tag{19}$$

where

$$\Pi_\mu = p_\mu - x'_\mu + 2W^1_{1\mu}, \quad \hat{\Pi}_\mu = p_\mu + x'_\mu - 2W^2_{1\mu}. \tag{20}$$

The algebra of constraints naturally splits into two sectors

$$\{d_{1\alpha}, d_{1\beta}\} = 2i\gamma^\mu_{\alpha\beta}\Pi_\mu, \quad \{d_{2\alpha}, d_{2\beta}\} = 2i\gamma^\mu_{\alpha\beta}\hat{\Pi}_\mu, \tag{21}$$

$$\{d_{1\alpha}, d_{2\beta}\} = 0, \quad \{d_{1\alpha}, \hat{\Pi}_\mu\} = 0, \quad \{d_{2\alpha}, \Pi_\mu\} = 0, \quad \{\Pi_\mu, \hat{\Pi}_\nu\} = 0. \tag{22}$$

To separate the constraints into first and second class we will write them in the light cone coordinates and divide the α, β spinor indices using the spinorial representation of the little group SO(8). The result is that the constraints $d^A_a = 0$, $(\Pi^+ \neq 0)$ are second class while the rest of the constraints $d^A_{\dot{a}} = 0, T = 0, \hat{T} = 0$ are first class. This fact allow us to count the number of degrees of freedom for the superstring given us the correct result, as expected. In what follows we will not need the details of this first class algebra as our aim is to construct a new effective gauge algebra. To that end we extend the original phase space $x^\mu, p_\mu, \theta^A_\alpha, p^A_\alpha$ by adding the fermionic variables S_a with the symplectic structure[1].

$$\{S_a, S_b\} = i\delta_{ab}. \tag{23}$$

Searching a solution for the condition (1)

$$\{\tilde{d}_a, \tilde{d}_a\} = 0, \tag{24}$$

in power series of S give us a very simple result. The solution is linear in S and yields

$$\tilde{d}_a = d_a + i\sqrt{2\Pi^+}S_a. \tag{25}$$

[1]In what follows we will work on the sector with index 1 and for simplicity we will remove this index from our equations. The other sector can be worked in the same way.

The next step consist in the deformation of the other first class constraints $d_{\dot{a}} = 0$, $T = 0$ to be consistent with the new $\tilde{d}_a$ constraints. Consider the case of $d_{\dot{a}}$. We need to find a solution to the condition

$$\{\tilde{d}_a, \tilde{d}_{\dot{a}}\} = 0. \tag{26}$$

The solution has the general form

$$\tilde{d}_{\dot{a}} = d_{\dot{a}} + A_{\dot{a}b} S_b + B_{\dot{a}[bc]} S_b S_c, \tag{27}$$

where

$$A_{\dot{a}b} = \frac{2i\gamma^i_{\dot{a}a}\Pi^i}{\sqrt{2\Pi^+}}, \quad B_{\dot{a}[ac]} = \frac{2\gamma^i_{b[a}\gamma^i_{c]\dot{a}}\theta'_b}{\Pi^+}. \tag{28}$$

The extended constraint is

$$\tilde{d}_{\dot{a}} = d_{\dot{a}} + \frac{2i\Pi^i}{\sqrt{2\Pi^+}}(\gamma^i S)_{\dot{a}} + \frac{2(\theta'\gamma^i S)(\gamma^i S)_{\dot{a}}}{\Pi^+}, \tag{29}$$

keeping in mind that the last term has to be antisymmetrized in undoted spinorial indices.

Now to find the extended constraint associated with T it is easy to proceed first to the extension of Π_μ defined in eq (20). To do that we need to solve the condition (1) for $\tilde{\Pi}_\mu$, i.e.,

$$\{\tilde{d}_a, \tilde{\Pi}_\mu\} = 0. \tag{30}$$

A simple check shows that the series in powers of the new variables S for $\tilde{\Pi}_\mu$ stops up to second order terms. The solution is

$$\tilde{\Pi}^\mu = \Pi^\mu + 4i\frac{(\theta'\gamma^\mu S)}{\sqrt{2\Pi^+}} + i\frac{S\gamma^\mu S}{\Pi^+}. \tag{31}$$

Using this solution we will define the new first class constraint $\tilde{T}$ in such a way that it will close in a Lie algebra with the rest of the new constraints

$$\tilde{T} = \frac{\tilde{\Pi}^2}{4\Pi^+} = -\frac{\Pi^-}{4} + \frac{\Pi^i\Pi^i}{4\Pi^+} + 2i\frac{\theta'_a S_a}{\sqrt{2\Pi^+}} + i\frac{S_a S'_a}{2\Pi^+} + 4i\frac{\Pi^i(\theta'\gamma^i S)}{(2\Pi^+)^{3/2}} - 2\frac{(\theta'\gamma S)^2}{(\Pi^+)^2}. \tag{32}$$

The new effective gauge algebra in the extended space is now

$$\{\tilde{T}, \tilde{T}\} = 0, \quad \{\tilde{d}_a, \tilde{d}_b\} = 0, \quad \{\tilde{d}_{\dot{a}}, \tilde{d}_a\} = 0, \tag{33}$$

$$\{\tilde{d}_a, \tilde{T}\} = 0, \quad \{\tilde{d}_{\dot{a}}, \tilde{d}_{\dot{b}}\} = -8i\tilde{T}\delta_{\dot{a}\dot{b}}. \tag{34}$$

The gauged GS first order action is

$$\tilde{S} = -\frac{1}{2}\int d^2\zeta\Big(\dot{x}^\mu p_\mu + \dot{\theta}^A_\alpha p^A_\alpha + \frac{i}{2}\dot{S}^A_a S^A_a - \lambda\tilde{T} - \hat{\lambda}\hat{\tilde{T}} - \lambda^A_\alpha \tilde{d}^A_\alpha\Big), \tag{35}$$

where we have included the two sectors. Its gauge symmetries are the worldsheet diffeomorphisms that are generated by $\tilde{T}$ and $\hat{\tilde{T}}$ and a new fermionic

gauge symmetry that is generated by $\tilde{d}_{\alpha}^{A}$. Of course the theory is not manifestly Lorentz covariant but the Lorentz invariance is guaranteed up to a BRST trivial transformation [1].

We have now 17 first class constraints by sector and as expected the model is equivalent to the original GS action and by construction has the same number of degrees of freedom. Two comments are in order. The first is that this embedding of the GS superstring is equivalent to the classical BM action. Its quantization can be performed along the same lines as the quantization of the BM model. Subtitles related to ordering ambiguities must be taken into account for a consistent quantization of this action. Secondly, as the BM model can be related to the pure spinor formalism via similarity transformations between the associated BRST charges, this model can also be related to the pure spinor formalism using the same sequence of similarity transformations between its associated BRST charges. The advantage of our perspective is that we have developed a gauge model in a completely systematic way starting from the plain GS superstring and consequently we have more control over any change in the embedding procedure that can be of help to relate GS and pure spinor formalisms in a more direct way. This procedure can also be of some help to better understand many aspects of pure spinor formalism like its geometrical interpretation, the path integral measure, or the underlying action.

From the other hand our new constraints (25,29,32) are the same as the ones obtained in [9]. It is quite surprising for us that the constraints are exactly the same. The two procedures are very different. In [9] the number of fermions is doubled and an interaction between them was introduced by hand. After fixing the semi-lightcone gauge and making a complicated Darboux transformation simplifying the Dirac bracket, the results of [9] coincides with the BFT embedding presented here. We will try to explain this relation in the next section by extracting more information about how the BFT embedding works.

4 Relation with the AK model

That the embedding approach to GS action has something to do with the interacting action proposed in [9] is at first sight very surprising. Here we will try to elaborate on this relation using a slightly modified approach to the conversion procedure. The arguments presented in this section does not apply to the case of a general constrained system but are valid in some special type of systems like the one considered here.

Lets start by noticing that another way to apply the BFT embedding is to seek for new extended coordinates $\tilde{x}^{\mu}, \tilde{p}_{\mu}, \tilde{\theta}_{\alpha}^{A}, \tilde{p}_{\alpha}^{A}$ such that they satisfy the conditions

$$\{\tilde{d}_{a}^{A}, \tilde{z}\} = 0, \tag{36}$$

where $\tilde{z}(x^{\mu}, p_{\mu}, \theta_{\alpha}, p_{\alpha}, S)$ is any of the phase space extended new coordinates or momenta. If we can solve these conditions then we can use the solutions to extend any observable of the original phase space to the new extended phase space. The procedure is as follows: suppose that we have a function in the

original phase space $A(z)$. First we write it as $A(\tilde{z})$, then substitute $\tilde{z}$ by the solution to (36) and find $\tilde{A}(z,S)$. For the GS superstring the solution to the conditions (36) are[2]

$$\tilde{x}^{\mu} = x^{\mu} - i\frac{(\theta\gamma^{\mu}S)}{\sqrt{2\Pi^{+}}}, \quad \tilde{\theta}_{a} = \theta_{a} - \frac{S_{a}}{\sqrt{2\Pi^{+}}}, \quad \tilde{\theta}_{\dot{a}} = \theta_{\dot{a}}, \tag{37}$$

for configuration space variables. For the bosonic momenta p_{μ} and the fermionic momenta p_{α} the solutions are

$$\tilde{p}^{\mu} = p^{\mu} + i\Big(\frac{\theta\gamma^{\mu}S}{\sqrt{2\Pi^{+}}}\Big)', \tag{38}$$

$$\tilde{p}_{\alpha} = p_{\alpha} - i\frac{(\gamma^{\mu}S)_{\alpha}}{\sqrt{2\Pi^{+}}}(\Pi_{\mu} - W_{1\mu} + P_{\mu}) + i(\gamma^{\mu}\theta)_{\alpha}P_{\mu} + C_{\alpha}, \tag{39}$$

where

$$P_{\mu} = 2i\frac{\theta'\gamma_{\mu}S}{\sqrt{2\Pi^{+}}} + i\frac{S\gamma_{\mu}S'}{2\Pi^{+}} + i\Big(\frac{\theta\gamma_{\mu}S}{\sqrt{2\Pi^{+}}}\Big)', \tag{40}$$

and

$$C_{a} = i\sqrt{2\Pi^{+}}S_{a}, \quad C_{\dot{a}} = \frac{2i\Pi^{i}}{\sqrt{2\Pi^{+}}}(\gamma^{i}S)_{\dot{a}} + \frac{2(\theta'\gamma^{i}S)(\gamma^{i}S)_{\dot{a}}}{\Pi^{+}}, \tag{41}$$

where the last term must be antisymmetrized with repect to the spinorial indices without dots. The new momenta $\tilde{p}_{\mu}$, $\tilde{p}_{\alpha}$ in (38, 39) are very similar to the Darboux transformation proposed in [9] to simplify the Dirac bracket but there are differences that we will explain below. What is perhaps more interesting is that the transformations (37) in configuration space can be used to obtain, from the GS action (11), the interacting AK action. Indeed, redefine $S_{a}/\sqrt{2\Pi^{+}}$ as ξ_{a} and use (37) to get

$$S = -\frac{1}{2}\int d^{2}\zeta\Big[\sqrt{-g}g^{ij}\Pi_{i}^{\mu}\Pi_{\mu j} + 2\varepsilon^{ij}\Pi_{i}^{\mu}(W_{j\mu}^{1} - W_{j\mu}^{2}) - 2\varepsilon^{ij}W_{i}^{1\mu}W_{j\mu}^{2}\Big], \tag{42}$$

where

$$W_{i}^{A\mu} = i\Theta^{A}\gamma^{\mu}\partial_{i}\Theta^{A}, \quad \Pi_{i}^{\mu} = \partial_{i}x^{\mu} - \sum_{A}W_{i}^{A\mu} - i\sum_{A}\partial_{i}(\theta^{A}\gamma^{\mu}\xi^{A}), \tag{43}$$

with $\Theta^{A} = \theta^{A} - \xi^{A}$ as in [9][3]. Notice that the effect of the substitution of the new configuration variables (37) in terms of the old ones in the original GS action (11) produces a deformation of the symplectic structure and a deformation of the original bosonic constraints. The first order action has now the form

$$S = -\frac{1}{2}\int d^{2}\zeta\Big(\dot{x}^{\mu}p_{\mu} + \dot{\theta}_{\alpha}^{A}\Delta_{\alpha}^{A} + \dot{\xi}_{a}^{A}\Xi_{a}^{A} - \lambda\tau - \hat{\lambda}\hat{\tau}\Big), \tag{44}$$

[2] This solutions are for the sector 1. The solution for the other sector has the same form.

[3] We denote by ξ the variable that are denoted as $\tilde{\theta}$ in [9] to avoid some possible confusion with our tilde variables.

where space-time momenta is

$$p_\mu = \Pi_{0\mu} - (W^1_{1\mu} - W^2_{1\mu}), \tag{45}$$

and the functions in the kinetic term are

$$\Delta^1_\alpha = -i(\gamma^\mu\xi^1)_\alpha p_\mu + i(p^\mu - \Pi^\mu_1 - W^{2\mu}_1)(\Theta^1\gamma_\mu)_\alpha, \tag{46}$$

$$\Delta^2_\alpha = -i(\gamma^\mu\xi^2)_\alpha p_\mu + i(p^\mu + \Pi^\mu_1 + W^{1\mu}_1)(\Theta^2\gamma_\mu)_\alpha, \tag{47}$$

and

$$\Xi^1_a = -i(\gamma^\mu\theta^1)_a p_\mu + i(p^\mu - \Pi^\mu_1 - W^{2\mu}_1)(\Theta^1\gamma_\mu)_a, \tag{48}$$

$$\Xi^2_a = -i(\gamma^\mu\theta^2)_a p_\mu + i(p^\mu + \Pi^\mu_1 + W^{1\mu}_1)(\Theta^2\gamma_\mu)_a. \tag{49}$$

$\tau, \hat{\tau}$ are the deformed bosonic constraints after the substitution of (37) in the original bosonic constraints T and $\hat{T}$. Defining, as usual, the fermionic momenta as the coefficient that multiply $\dot{\theta}$ in (42), we find the fermionic constraints

$$D^A_\alpha = p^A_\alpha - \Delta^A_\alpha, \tag{50}$$

that correspond to the constraints d^A_α of the original GS action. Integration by parts in the kinetic term produces the action

$$S = -\frac{1}{2}\int d^2\zeta\Big(\dot{x}^\mu(p_\mu - P_\mu) + \dot{\theta}^A_\alpha(p^A_\alpha - P^A_\alpha) + i\Pi^+\dot{\xi}^A_a\xi^A_a - \lambda\tau - \hat{\lambda}\hat{\tau} - \lambda^A_\alpha D^A_\alpha\Big), \tag{51}$$

where

$$P_\mu = -i(\theta\gamma^\mu\xi)'_1 + i(\theta\gamma^\mu\xi)'_2, \tag{52}$$

and $()_1$ denotes variables of the sector 1 and , $()_2$ for sector 2. The fermionic momenta redefinition is

$$P_a = ix'^+\xi_a + R_a, \quad P_{\dot{a}} = (\gamma^i\xi)_{\dot{a}}\Big(ix'_i + (\theta\gamma_i\xi)_2\Big) + R_{\dot{a}}, \tag{53}$$

where

$$R_\alpha = (\theta\gamma^\mu)_\alpha\Big(-2(\xi\gamma_\mu\theta') + \xi\gamma_\mu\xi' + (\theta\gamma_\mu\xi)'\Big) + (\xi\gamma^\mu)_\alpha\Big(3(\theta\gamma_\mu\theta') - 2(\theta\gamma_\mu\xi')\Big). \tag{54}$$

The redefinition of the space time momenta are the same as the redefinition that we have used in the BFT embedding but the redefinitions associated with the fermionic momenta are slightly different. The reason is that the fermionic constraints D^A_α are not the same as the original fermionic constraints d_α *after* the substitution of the new coordinates (37). Now it is easy to check that the field redefinitions

$$p_\mu \to p_\mu - P_\mu, \quad p^A_\alpha \to p^A_\alpha - P_\alpha, \quad \xi_a = S_a/\sqrt{2\Pi^+}, \tag{55}$$

produces the first order action (35) obtained by the BFT embedding. The efficient way, just presented to analyze the constraint structure of the action

(42) is inspired in the Faddeev-Jackiw method that is equivalent to the Dirac method for a wide class of constrained systems [11]. So we have obtained the interacting model [9] from the BFT embedding approach using the configuration variables $\tilde{x}^\mu, \tilde{\theta}^A_\alpha$ obtained by solving the embedding condition (36). Notice that this procedure does not work for a general constrained system. The fact that the GS action is linear in the velocities of the fermionic variables and that the solution to (36) depends only on the configuration variables, after the redefinition that relates S with ξ are crucial ingredients in the construction of a Lagrangian action compatible with the dynamics of the original Hamiltonian action.

5 Conclusions

We have worked out the BFT embedding of the GS formalism using light-cone variables. We have shown that the correction to the second class constraints and the embedding of the first class algebra requires only terms up to second order in the new fermionic variables and its derivatives. The embedding can be performed in a completely systematic way and we do not have to fix the gauge at any stage of our embedding procedure not to add ad-hoc variables other than the ones needed by the BFT embedding approach. The gauged GS action that results from our analysis is equivalent to the BM model proposed recently in [1]. It also explain some aspects of the rationale under the construction of a related model worked in [9].

The systematics behind our approach can be used to study the relation of the BRST charges and associated cohomologies between this model and the pure spinor formalism. We also have more control on the gauge fixing procedure. It is also a better point of departure to study the supermembrane and other topics not yet well understood in the pure spinor formalism, like the associated action and the measure of the path integral. It could be also of interest to explore the idea of non-Abelian [12] conversion to simplify the relation between the Berkovits pure spinor formalism and the GS embedding. We will return to these aspects elsewhere.

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Appendix

The notation used in the paper is the following: for the GS action (11), $\varepsilon^{01} = 1$, θ^A_α, is an SO(9,1) spinor with $A = 1, 2$ supersymmetries and $\alpha = 1, 2, ...16$ components. They are real Mayorana-Weyl spinors of the same chirality. x^μ

$\mu = 0, 1, 2...9$ are the space-time configuration variables. The worldsheet coordinates are $\zeta = (t, \sigma)$ and the derivatives with respect to time will be denoted by a dot and with respect to sigma by a prime. The γ^μ matrices are 16×16 Dirac matrices, real and symmetric. The convention for derivatives are left derivatives and then

$$\{\theta^A_\alpha, p^B_\beta\} = -\delta_{\alpha\beta}\delta^{AB}, \tag{56}$$

This convention fixes the order of $\dot\theta$ and p in the kinetic term of the first order action.

Light-cone coordinates: We split the spinorial index α according to the the SO(8) chiral and anti-chiral components a and $\dot a$. Space-time indices decompose according to $\gamma^\pm = \gamma^0 \pm \gamma^9$, $x^\pm = x^0 \pm x^9$... and γ^i, x^i, i=1,2,...8 for the other set of vector components. The Dirac algebra decomposes according to

$$\gamma^+_{\dot a \dot b} = -2\delta_{\dot a \dot b}, \quad \gamma^-_{ab} = -2\delta_{ab}, \quad \gamma^i_{\dot a a}\gamma^j_{\dot a b} + \gamma^j_{a \dot a}\gamma^i_{\dot a b} = 2\delta^{ij}\delta_{ab}, \tag{57}$$

and

$$\gamma^i_{a\dot b}\gamma^i_{c\dot d} + \gamma^i_{a\dot d}\gamma^i_{c\dot b} = 2\delta^{ac}\delta_{\dot b \dot d}, \tag{58}$$

with $\gamma_{\dot a a}$ symmetric. We use repeatedly through the text the Fierz identity

$$(\gamma_{(\alpha\beta})^\mu(\gamma_{\gamma)\delta})_\mu = 0. \tag{59}$$

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